

# ABCD matrices for non-rectilinear propagation of light

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In the “Notes and Discussion” article Ref. [1] the authors present a very interesting, although purely mathematical, matrix representation of some special polynomials. Here we apply this mathematical formalism to a physical case, namely non-rectilinear propagation of rays of light in arbitrary non-homogeneous media.

Rays of light propagate along rectilinear trajectories in air. Therefore, at the generic position  $x$  a ray (represented by the linear function  $f(x) = a + bx$ ) is completely determined by a pair numbers solely:  $f(x)$  and  $f'(x)$ . Such a pair may be represented in a vector-like form as follows:

$$\mathbf{f}(x) = \begin{bmatrix} f(x) \\ f'(x) \end{bmatrix}. \quad (1)$$

The simple linear relation existing between  $\mathbf{f}(x_1)$  and  $\mathbf{f}(x_2)$  at two arbitrarily chosen positions  $x_1$  and  $x_2 = x_1 + L$ , with  $L > 0$ , is usually written in optics textbooks [2–5] in the following matrix form:

$$\begin{bmatrix} f(x_2) \\ f'(x_2) \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(x_1) \\ f'(x_1) \end{bmatrix}. \quad (2)$$

The  $2 \times 2$  matrix in Eq.(2) is known as the *ABCD* matrix of the optical system (free space, in the present case) and fully characterizes the propagation of rays of light in it.

Literature is rich of examples of *ABCD* matrices for more complicated optical systems as lenses, planes and curved dielectric interfaces, mirrors, inhomogeneous media with a quadratic index profile et cetera, and combinations thereof [6, 7].

For complex inhomogeneous media, the trajectory of a ray of light is more complicated than a straight line and cannot be represented anymore by a linear function, except then for a very short distance  $x_2 - x_1$ . If this is not the case, then one may either subdivide the bulk material of macroscopic thickness  $L$  in  $N \gg 1$  slices of infinitesimal thickness  $dL = L/N$  and use  $N$  different *ABCD* matrices (one for each slice) [8], or use the alternative approach that will be the subject of this note.

This method is based on the observation that the *ABCD* matrix formalism can be viewed as a linearization of the trajectory of a ray of light around the initial point  $x_1$  [9, 10]. Such linearization, namely a Taylor expansion truncated up to and including first order terms, is physically meaningful only for  $x_2$  close enough to  $x_1$ :  $x_2 \simeq x_1 + L/N$ . But what if  $x_2$  is no longer close to  $x_1$ ? Does the first order Taylor expansion break down? If so, can such expansion be suitably extended? If higher order terms must be retained, what is their physical meaning? To answer these questions, we put on rigorous basis this linearization procedure showing that a  $2 \times 2$  *ABCD* matrix is a principal sub-matrix [11] of an effective  $\infty \times \infty$  matrix describing the full nonlinear dynamics of a curvilinear ray of light. In doing so, we feel that re-expressing and generalizing a standard result in classical optics (namely, the *ABCD* matrix formalism) in the more familiar language of linear algebra and basic calculus would make this topic, usually reserved to graduate students in optics, accessible to a more broad audience.

To begin with, let us first re-derive Eq.(2) for an arbitrary linear function  $f(x)$  that now we write in the following manner:

$$f(x) = a_0 + a_1 x \equiv a_0^i + a_1^i (x - x_i), \quad (3)$$

where  $x_i$  is an arbitrary point belonging to the domain of the function  $f(x)$ . If we choose  $x_i = 0$  then we retrieve the previous expression  $f(x) = a + bx$  with  $a_0 = a$  and  $a_1 = b$ . However, for  $x_i \neq 0$ , the last equality in Eq.(3) gives:

$$a_0^i = a_0 + a_1 x_i = f(x_i), \quad (4a)$$

$$a_1^i = a_1 = f'(x_i), \quad (4b)$$

where, for the sake of simplicity, we have introduced the notation  $a_k^i = a_k(x_i)$ .

Since the point  $x_i$  is arbitrarily chosen, we can choose a different point  $x_j > x_i$  and write:

$$f(x) = a_0^i + a_1^i (x - x_i) = a_0^j + a_1^j (x - x_j). \quad (5)$$

By equating the factors with the same powers of  $x$  at the second and third terms in the equation above, we

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obtain  $a_0^j = a_0^i + (x_j - x_i)a_1^i$  and  $a_1^j = a_1^i$ . This can be rewritten in the following matrix form:

$$\begin{bmatrix} a_0^j \\ a_1^j \end{bmatrix} = \begin{bmatrix} 1 & x_j - x_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_0^i \\ a_1^i \end{bmatrix}, \quad (6)$$

This result is equivalent with the one written in Eq. (2), if we identify  $a_0^j = f(x_2)$ ,  $a_1^j = f'(x_2)$ ,  $a_0^i = f(x_1)$  and  $a_1^i = f'(x_1)$ . The formal derivation of Eq.(2) via the steps (3-6) it is highly redundant for the linear-function case. However, it has the virtue to be generalizable to the case of non-rectilinear ray propagation.

Now, in order to describe a ray that propagates in an inhomogeneous medium we need a generic smooth non-linear function  $f(x)$  which can be expanded in a Taylor series around  $x = 0$

$$f(x) = a_0 + a_1x + a_2x^2 + \dots \quad (7)$$

For any  $x_i \in \mathbb{R}$  we can write  $x = x - x_i + x_i$  and insert this relation into Eq.(7) to obtain

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_i + x_i) + a_2(x - x_i + x_i)^2 + \dots \\ &= \sum_{n=0}^{\infty} a_n^i (x - x_i)^n, \end{aligned} \quad (8)$$

where the  $a_n^i$  coefficient are given by:

$$a_0^i = f(x_i), \quad (9a)$$

$$a_n^i = \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=x_i} = \sum_{k=n}^{\infty} \binom{k}{n} a_k x_i^{n-k}. \quad (9b)$$

Now we can repeat the same reasoning that lead to Eq.(8) and write the following equality:

$$\sum_{n=0}^{\infty} a_n^i (x - x_i)^n = \sum_{n=0}^{\infty} a_n^j (x - x_j)^n, \quad (10)$$

where again  $x_i$  and  $x_j$  are two arbitrarily points on the real axis with  $x_j > x_i$ .

By expanding both sides of this equation with the help of the Newton's binomial formula one gets

$$\begin{aligned} \sum_{n=0}^{\infty} a_n^i \sum_{k=0}^n \binom{n}{k} x^k (-x_i)^{n-k} \\ = \sum_{n=0}^{\infty} a_n^j \sum_{k=0}^n \binom{n}{k} x^k (-x_j)^{n-k}. \end{aligned} \quad (11)$$

This expression can be turned into a recursive relation by equating terms with the same power of  $x$ . Then, for  $k = 0$  we have:

$$\sum_{n=0}^{\infty} (-1)^n a_n^i x_i^n = \sum_{n=0}^{\infty} (-1)^n a_n^j x_j^n, \quad (12)$$

which can be rewritten, after isolating the  $n = 0$  term, as:

$$a_0^j = a_0^i + \sum_{n=1}^{\infty} (-1)^n (a_n^i x_i^n - a_n^j x_j^n). \quad (13)$$

For  $k = 1$  the same procedure gives

$$a_1^j = a_1^i + \sum_{n=2}^{\infty} (-1)^{n-1} (a_n^i x_i^{n-1} - a_n^j x_j^{n-1}). \quad (14)$$

This method can be iterated by choosing  $k = 2, 3, \dots, n$  to generate the following recursive relation:

$$a_k^j = a_k^i + \sum_{n=k+1}^{\infty} \binom{n}{k} (-1)^{n-k} (a_n^i x_i^{n-k} - a_n^j x_j^{n-k}), \quad (15)$$

with  $k = 0, 1, \dots, n$ . The equation above can be seen as a linear algebraic system relating the variables  $a_n^i$  to the quantities  $a_n^j$ . This result can be then written in matrix form. Let us introduce the following notation

$$b_n^j(k) = \binom{n}{k} (-1)^{n-k} x_j^{n-k}. \quad (16)$$

Note that  $b_k^j(k) = 1$ . If we now assume to truncate the summation in Eq.(15) to a finite value  $N$  in order to be able to represent these results (formally described by an  $\infty \times \infty$  matrix) with a finite dimensional matrix as follows:

$$\sum_{n=k}^N b_n^j(k) a_n^j = \sum_{n=k}^N b_n^i(k) a_n^i. \quad (17)$$

By then defining  $\mathbf{a}^j = (a_0^j, a_1^j, \dots, a_N^j)$  and  $\mathbf{a}^i = (a_0^i, a_1^i, \dots, a_N^i)$ , it is possible to rewrite the previous equation in matrix form as follows:

$$\begin{bmatrix} b_0^j(0) & b_1^j(0) & b_2^j(0) & \cdots \\ 0 & b_1^j(1) & b_2^j(1) & \cdots \\ 0 & 0 & b_1^j(2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0^j \\ a_1^j \\ a_2^j \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0^i(0) & b_1^i(0) & b_2^i(0) & \cdots \\ 0 & b_1^i(1) & b_2^i(1) & \cdots \\ 0 & 0 & b_1^i(2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0^i \\ a_1^i \\ a_2^i \\ \vdots \end{bmatrix}. \quad (18)$$

If we call  $\mathbf{B}(j)$  the matrix on the left-hand-side of the previous equation and  $\mathbf{B}(i)$  the one on the right-hand-side, the previous relation can be written in the following compact form:

$$\mathbf{B}(j)\mathbf{a}^j = \mathbf{B}(i)\mathbf{a}^i, \quad (19)$$

where again the shorthand notations  $\mathbf{B}(i) = \mathbf{B}(x_i)$  and  $\mathbf{B}(j) = \mathbf{B}(x_j)$  are used for the sake of clarity. Solving for  $\mathbf{a}^j$  by multiplying on the left both sides of the previous equation by  $\mathbf{B}(j)^{-1}$  and defining  $\mathbf{A} = \mathbf{B}^{-1}(j)\mathbf{B}(i)$ , we can write the relation between the vectors  $\mathbf{a}^j$  and  $\mathbf{a}^i$  as

$$\mathbf{a}^j = \mathbf{A}\mathbf{a}^i. \quad (20)$$

The matrix  $\mathbf{A}$  is our sought *generalized ABCD* matrix, whose expression is the following:

$$\begin{aligned} \mathbf{A} &= \mathbf{B}^{-1}(j)\mathbf{B}(i) \\ &= \begin{bmatrix} 1 & L & L^2 & L^3 & \cdots \\ 0 & 1 & 2L & 3L^2 & \cdots \\ 0 & 0 & 1 & 3L & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &\equiv \mathbf{A}(L), \end{aligned} \quad (21)$$

where  $L = x_j - x_i$ . Note that this matrix contains the usual (i.e. linear) *ABCD* matrix defined in Eq.(2) as the first  $2 \times 2$  principal sub matrix. This sub matrix verifies the following equality:

$$\begin{bmatrix} 1 & L_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & L_1 + L_2 \\ 0 & 1 \end{bmatrix}, \quad (22)$$

namely  $A(L_1)A(L_2) = A(L_1 + L_2)$ . The next step is to extract the principal  $3 \times 3$  submatrix from Eq. (21), i.e. to consider the first nonlinear order of the approximation of the function  $f(x)$ . In this case, the relation  $A(L_1)A(L_2) = A(L_1 + L_2)$  still holds, as can be checked by direct calculation:

$$\begin{aligned} &\begin{bmatrix} 1 & L_1 & L_1^2 \\ 0 & 1 & 2L_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L_2 & L_2^2 \\ 0 & 1 & 2L_2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & L_1 + L_2 & (L_1 + L_2)^2 \\ 0 & 1 & 2(L_1 + L_2) \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad (23)$$

This equation has a straightforward physical meaning: it illustrates the fact that propagation across two consecutive distances  $L_1$  and  $L_2$  can be described as a single propagation along the distance  $L_1 + L_2$ . From a mathematical point of view, this is a signature of the semigroup property of our generalized *ABCD* matrices. With the help of a suitable mathematical software for algebraic manipulation, it is not difficult to verify via explicit  $N \times N$  matrix multiplications, that Eq.(23) is valid for arbitrary  $N$ . Thus, by iteration, one can easily convince oneself that the matrix  $A$  satisfies the following relation [3]:

$$\prod_{n=1}^N A(L_n) = A\left(\sum_{n=1}^N L_n\right). \quad (24)$$

This result has a straightforward interpretation: the propagation of the function through the total distance  $L_1 + L_2 + \cdots + L_N$  can be achieved by consecutive propagation across the distances  $L_1, L_2, \dots, L_N$ .

This analogy is not accidental. A closer inspection to Eq.(20) reveals in fact that this equation tells us how the value of the function  $f(x)$  in a point  $x_j$  can be calculated knowing the value of the same function in a point  $x_i < x_j$ . With this in mind, we can calculate the derivative of  $f(x)$  as follows:

$$\begin{aligned} \frac{df(x)}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{x_j \rightarrow x_i} \left( \frac{\mathbf{a}^j - \mathbf{a}^i}{x_j - x_i} \right) \\ &= \lim_{L \rightarrow 0} \left( \frac{\mathbf{A} - \mathbf{I}}{L} \right) \mathbf{a}^i \equiv \mathbf{D}\mathbf{a}^i, \end{aligned} \quad (25)$$

where we choose  $\Delta x = x_j - x_i \equiv L$  so that we can express  $f(x + \Delta x)$  as  $\mathbf{a}^j$  and  $f(x)$  as  $\mathbf{a}^i$ . Note that this does not cause any loss of generality, since the definition of derivative involves only the concept of neighboring points and, as discussed previously, the quantities  $\mathbf{a}^i$  and  $\mathbf{a}^j$  represent the value of the function  $f(x)$  in two arbitrary neighboring points. Note moreover that in the last equality we used Eq.(20) to write  $\mathbf{a}^j$  as a function of  $\mathbf{a}^i$ . Here,  $\mathbf{D}$  is the matrix representation of the differential operator  $d/dx$  [1]

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (26)$$

It is not difficult to see, via an explicit calculation, that the generalized  $ABCD$  matrix is related to the differential operator by the following formula:

$$\mathbf{A}(L) = \sum_{k=0}^{\infty} \frac{(\mathbf{L}\mathbf{D})^k}{k!} = e^{\mathbf{L}\mathbf{D}}. \quad (27)$$

From didactic point of view it is very gratifying to see that we were able to reproduce the already known re-

sult for the translation operator acting upon an arbitrary function. In fact, Eq. (27) gives an actual *physical* representation of the well-known translation operator  $e^{L(d/dx)}$  [12], such that:

$$e^{L\frac{d}{dx}}f(x)|_{x=0} = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} L^k = f(L). \quad (28)$$

In our case, in fact, the action of the  $\mathbf{A}$  matrix completely defines the vector  $\mathbf{F}$  (whose elements are the all  $a_k^i$ ) at the point  $x_2$  knowing the expression of the  $\mathbf{F}$  vector at the point  $x_1$ , i.e.

$$\mathbf{F}(x_2) = e^{\mathbf{L}\mathbf{D}}\mathbf{F}(x_1) = \mathbf{A}(L)\mathbf{F}(x_1). \quad (29)$$

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